ISSN: 2178-7727

Common Fixed Points of Two Weakly Compatible Mappings in Fuzzy Metric Spaces

Amit Kumar[1] and Khushboo Singh[2*]
[1,2]Department of Mathematics, Banasthali Vidyapith, Newai, Rajasthan 304022, India.
Email: [1]amitsu48@gmail.com, [2*]mohiyang79@gmail.com

Abstract

In this paper, we investigate the notions of convexity, strict convexity, and normal structure in the framework of fuzzy metric spaces in the sense of George and Veeramani. These concepts are developed to extend classical geometric ideas from metric spaces to fuzzy settings. Using these structures, we establish the existence of a common fixed point for a pair of weakly compatible non-expansive self-mappings defined on compact convex subsets of strictly convex fuzzy metric spaces. The main result relies on the existence of normal structure and employs Zorn's lemma to ensure minimal invariant subsets. Our theorem generalizes Takahashi's fixed point theorem from convex metric spaces to fuzzy metric spaces and extends it from a single mapping to a pair of mappings. The results obtained unify and extend several known fixed point theorems in fuzzy and probabilistic metric spaces, providing a broader framework for further developments in nonlinear analysis within fuzzy environments.

2020 Mathematics Subject Classifications: 47H09, 47H10, 47H30, 54E40

Keywords: Fixed Points, Non-expansive Mappings, Weakly Compatible Mappings, Fuzzy Metric Spaces.

INTRODUCTION

Convexity is an important and widely studied concept in analysis and geometry. It is closely connected with many other areas of mathematics and also appears in several problems in the natural sciences. Because of these connections, convexity has been studied in many different types of spaces.

The main purpose of this paper is to introduce and study the concepts of convex, strictly convex, and normal structures for subsets of fuzzy metric spaces in the sense of George and Veeramani [2]. After establishing these notions, we prove a common fixed point theorem for a pair of weakly compatible non-expansive self-mappings defined on strictly convex fuzzy metric spaces that possess a normal structure. Our result follows the ideas of the classical fixed point theorem of Takahashi [5] and extends similar developments that have been made in probabilistic metric spaces and intuitionistic fuzzy metric spaces.

The paper is organized as follows. In Section 2, we recall some basic concepts related to fuzzy metric spaces and introduce the notions of compactness and fuzzy diameter. In Section 3, we define convex and strictly convex structures in fuzzy metric spaces and introduce the ideas of normal structure and non-expansive mappings, along with the concept of weakly compatible mappings. In Section 4, we show that compact convex subsets of strictly convex fuzzy metric spaces possess a normal structure. Finally, in Section 5, we present our main result, which establishes a common fixed point theorem for two weakly compatible non-expansive mappings on compact convex subsets of strictly convex fuzzy metric spaces.

ISSN: 2178-7727

2. PRELIMINARIES

We begin with the standard notion of a continuous t-norm.

Definition 2.1. A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is called a *continuous t-norm* if it satisfies:

(a) $*$ is commutative and associative;

(b) $*$ is continuous;

(c) $a * 1 = a$ for all $a \in [0,1]$;

(d) $a \leq c$ and $d \leq d$ imply $a * b \leq c * d$ for all $a, b, c, d \in [0,1]$.

Some examples are $a * b = min\{a, b\}$ and $a * b = ab$.

In this paper, we consider fuzzy metric spaces defined in the sense of George and Veeramani [2].

Definition 2.2. [2]**.** A *fuzzy metric space* is a triple $(X, M, *)$ where X is a non-empty set, $*$ is a continuous t-norm, and $M : X \times X \times (0, \infty) \to (0,1]$ satisfies, for all $x, y, z \in X \, and \, s, t > 0$:

(FM1) $M(x, y, t) > 0$;

(FM2) $M(x, y, t) = 1$ if and only if $x = y$;

(FM3) $M(x, y, t) = M(y, x, t)$;

(FM4*)* $M(x\, z, t + s) \geq M(x, y, t) * M(y, z\, s)$;

(FM5) $M(x, y,) : (0, \infty) \to (0,1]$ is continuous.

The function M is called a *fuzzy metric* on X, and $M(x, y, t)$ is interpreted as the degree of nearness between x and y with respect to the time parameter t.

Remark 1. As usual, from (FM4) one obtains that $M(x, y,)$ is non-decreasing in *t* for each fixed $x, y \in X$.

The most common example of a fuzzy metric space that arises from an ordinary metric space is as follows:

Example 2.3. Consider the metric space (X, d) and let $*$ be the product t-norm $a * b = ab$. Then for $x, y \in X$ and $t > 0$ set

$$M(x, y, t) = e^{-d(x,y)t}$$

is fuzzy metric and the triplet $(X, M,*)$ is a fuzzy metric space.

We recall the topology generated by a fuzzy metric, as found in standard references.

Definition 2.4. [1] Let $(X, M,*)$ be a fuzzy metric space. For$x \in X, r \in (0,1)$, and $t > 0$, the open ball with center x and radius r at time t is

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

ISSN: 2178-7727

It is well known that the family of all open balls forms a base for a Hausdorff topology on X. We denote this topology by τ_M.

Definition 2.5. A subset $K \subset X$ is called *compact* if it is compact with respect to the topology τ_M; that is, every open cover of K has a finite subcover.

We will use the following characterization in terms of closed sets.

Lemma 2.6. Consider a fuzzy metric space $(X, M, *)$ and let $K \subset X$. Then K is compact if and only if for every collection $\{F_a : a \in A\}$ of closed subsets of X with $F_a \subset K$ for each $a \in A$, the condition

$$\bigcap_{a \in A} F_a = \emptyset$$

implies that there exist $a_1, \dots, a_n \in A$ such that $\bigcap_{i=1}^{n} F_{a_i} = \emptyset$

We next introduce a fuzzy analogue of diameter.

Definition 2.7. [3] Let $(X, M, *)$ be a fuzzy metric space and $A \subset X$. For $t > 0$, define

$$d_A(t) = inf_{x,y \in A} \sup_{0<\varepsilon<t} M(x, y, \varepsilon)$$

The constant

$$d_A = \sup_{t>0} d_A(\mathrm{t})$$

is called the *fuzzy diameter* of the set A. We say that A is *strongly fuzzy bounded* if $d_A = 1$.

Remark 2. Every compact subset A of a fuzzy metric space is strongly fuzzy bounded. This follows because the function M is continuous and the *set* $A \times A \times (0, t]$ is compact for each $t > 0$.

3. CONVEX STRUCTURE AND NORMAL STRUCTURE ON FUZZY METRIC SPACES

We now discuss a notion of convex structure in fuzzy metric spaces. This idea is similar to the convex structure defined by Takahashi in metric spaces [5]. Later, this concept was extended to fuzzy metric spaces by $\grave{J}esi\grave{c}$ [3] in the following way:

Definition3.1. [3] Let $(X, M, *)$ be a fuzzy metric space. We say that X possesses a *convex structure* if there exists a mapping

$$S: X \times X \times [0,1] \to X$$

Such that

(I) $S(x, y, 0) = \mathrm{y}$ and $S(x, y, 1) = x$ for all $x, y \in X$;
(II) For all $x, y, z \in X, k \in (0,1)$ and $t > 0$

$$M(S(x, y, k), z, 2t) \geq M\left(x, z, \frac{t}{k}\right) * M\left(y, z, \frac{t}{1-k}\right) \qquad (1)$$

A fuzzy metric space $(X, M, *)$ together with such an S is called a *convex fuzzy metric space*.

ACTA SCIENTIAE

ISSN: 2178-7727

A normed linear space$(X, \|.\|)$ can be viewed as a convex fuzzy metric space, as shown in the following example:

Example3.2. Let $(X, \|.\|)$ be a normed linear space. Define

$$M(x,y,t) = e^{-\frac{\|x-y\|}{t}}, x, y \in X, t > 0,$$

and take $*$ as the product t-norm $a * b = ab$.

Now define

$S(x,y,k) = kx + (1-k)y, k \in (0,1).$

It can be easily checked that condition (1) holds for all $x, y, z \in X, k \in (0,1)$ and $t > 0$. Hence $(X, M, *, S)$ is a convex fuzzy metric space.

Example 3.3. Let $d(x,y) = x - y$ and

$$M(x,y,t) = \frac{t}{t + \| x - y \|}$$

with $*$ taken as the minimum *t*-norm. Define

$$S(x,y,k) = kx + (1-k)y.$$

Then one checks that (1) holds for all $x, y, z \in X, k \in (0,1)$ and $t > 0, so$ $(X, M, *, S)$ is a convex fuzzy metric space.

Definition 3.4. Let $(X, M, *, S)$ be a convex fuzzy metric space. A subset $A \subset X$ is said to be convex if for every $x, y \in A$ and $k \in [0,1]$ we have $S(x,y,k) \in A$.

Lemma3.5. [3] Let $(X, M, *, S)$ be a convex fuzzy metric space and $\{K_\alpha\}_{\alpha \in D}$ a family of convex subsets of X. Then their intersection

$$K = \bigcap_{\alpha \in D} K_\alpha$$

is convex.

Proof. If $x, y \in K$, then $x, y \in K_\alpha$ for each $\alpha \in D$, so $S(x,y,k) \in K_\alpha$ for all $\alpha \in D$ and all $k \in [0,1]$. Hence $S(x,y,k) \in K$.

We now give a strict convexity condition as follows:

Definition 3.6. A convex fuzzy metric space $(X, M, *, S)$ is said to be *strictly convex* if for any two distinct points $x, y \in X$, any $k \in (0,1)$, and every $t > 0$, the point $z = S(x,y,k)$ is the unique element of X such that

$M\left(x, y, \frac{t}{k}\right) = M(z,y,t)$ and $M\left(x, y, \frac{t}{1-k}\right) = M(x,z,t)$

For the main result, we will also use a stronger form of convexity inequality.

Lemma 3.7. Let $(X, M,*, S)$ be a convex fuzzy metric space satisfying the following condition: for all $x, y, z \in X, k \in (0,1)$ and $t > 0$,

$$M(S(x, y, k), z, t) > min\{M(z, x, t), M(z, y, t)\} \quad (2)$$

Assume that there exists a point $z \in X$ for which

$M(S(x, y, k), z, t) = min\{M(z, x, t), M(z, y, t)\}$ for every $t>0$.

Then it follows that the point $S(x, y, k)$ must coincide with either x or y, that is,

$$S(x, y, k) \in \{x, y\}.$$

Proof. If (2) holds and equality occurs for some $z \in X$ and all $t > 0$, then necessarily $k = 0 \; or \; k = 1$. Indeed, if $k \in (0,1)$, (2) would give a strict inequality. Hence $S(x, y, 0) = y \; or \; S(x, y, 1) = x$, so $S(x, y, k) \in \{x, y\}$

Lemma 3.8. [3] Let $(X, M,*, S)$ be a strictly convex fuzzy metric space satisfying (2). Then for every $x, y \in X$ with $x \neq y$ there exists $k \in (0,1)$ such that $S(x, y, k) \notin \{x, y\}$.

Proof. Assume, on the contrary, that for all $k \in [0,1]$ we have $S(x, y, k) \in \{x, y\}$. Using the defining equalities for strict convexity and the properties of M, one obtains $M(x, y, t) = 1$ for all $t > 0$, so $x = y$, which is a contradiction.

We now give diametral points and normal structure.

Definition 3.9. [4] Let $(X, M,*)$ be a fuzzy metric space and let $A \subset X$ be a set with $d_A > 0$. A point $x \in A$ is called a *diametral point* (with respect to M) if, for every $t>0$, the following holds:

$$d_A(t) = inf_{y \in A} \, sup_{0<\epsilon<t} \, M(x, y, \varepsilon)$$

Definition 3.10. Let $(X, M,*, S)$ be a convex fuzzy metric space. We say that X has a normal structure if every closed, strongly fuzzy bounded, and convex subset $Y \subset X$ containing at least two different points has a point $x \in Y$ which is not diametral. In other words, there exists $t_0 > 0$ such that

$$inf_{y \in Y} \, sup_{0<\epsilon<t_0} M(x, y, \varepsilon) > d_Y(t_0)$$

Definition 3.11. Let $(X, M,*, S)$ be a convex fuzzy metric space and let $Y \subset X$. The *closed convex hull* of Y, denoted by $conv(Y)$, is defined as the smallest closed and convex subset of X that contains Y. In other words, it is the intersection of all closed convex subsets of X that contain Y.

The existence of $conv(Y)$ is clear, since X is closed and convex and contains Y . By the previous lemma, the intersection of convex sets is convex, and intersections of closed sets are closed.

Definition 3.12. Let $(X, M,*)$ be a fuzzy metric space and let $f : X \to X$ be a mapping. The mapping f is said to be *non-expansive* (with respect to M) if

$M(fx, fy, t) \geq M(x, y, t)$ for all $x, y \in X$ and $t > 0$.

Definition 3.13. Let $f, g : X \to X$ be mappings on a fuzzy metric space $(X, M,*)$. The mappings f and g are said to be *weakly compatible* if they commute at their coincidence points. That is, if there exists $x \in X$ such that

ISSN: 2178-7727

$fx = gx$, then $f(gx) = g(fx)$.

A point $x \in X$ for which $fx = gx$ is called a *coincidence point* of f and g. A point $x \in X$ is called a *common fixed point* of f and g if $fx = x$ and $gx = x$.

4. NORMAL STRUCTURE FOR COMPACT CONVEX SETS

In this section, we first prove that compact convex subsets of strictly convex fuzzy metric spaces satisfying condition (2) possess a normal structure.

Lemma 4.1. *Consider* $(X, M, *, S)$ *is a strictly convex fuzzy metric space* satisfying (2). If K is a *non-empty compact convex subset of* X, *then* K *have a normal structure*.

Proof. Suppose, on the contrary, that K does not possess a normal structure. Then every point of K is diametral. Let $d_K(\mathrm{t})$ denote the fuzzy diameter of K. Since K is compact and the function

$$\emptyset(\boldsymbol{x}) = \inf_{y\in K} \sup_{0<\varepsilon<t} M(x, y, \varepsilon)$$

is continuous on K, there exists $x_0 \in K$ such that

$$\emptyset(x_0) = d_K(t)$$

Thus x_0 is a diametral point of K.

Because K contains at least two points, there exists $y_0 \in K$ with $x_0 = y_0$. Since K is convex, for any $k \in (0,1)$ the point

$$z = S(x_0, y_0, k)$$

belongs to K.

From the strong convexity condition (2), we have for every $w \in K$ and $t > 0$,

$$M(S(x_0, y_0, k), w, t) > \min\{M(w, x_0, t), M(w, y_0, t)\}.$$

Taking the infimum over $w \in K$ and the supremum over $0 < \varepsilon < t$, it follows that

$\inf_{w\in K} \sup_{0<\varepsilon<t} M(z, w, \varepsilon) > d_K(t)$

Hence z is not a diametral point of K, which contradicts the assumption that every point of K is diametral.

Therefore K must contain a point which is not diametral. Hence K possesses a normal structure.

We also need that closed balls are convex under the strong convexity condition.

Lemma 4.2. Let $(X, M, *, S)$ be a *convex fuzzy metric space satisfying* (2). Then for each $x \in X, r \in (0,1)$ and $t > 0$, the closed ball

$$B(x, r, t) = \{y \in X: M(x, y, t) \geq 1 - r\}$$

is a convex subset of X.

ISSN: 2178-7727

Proof. Let $a, b \in B(x, r, t)$, so $M(a, x, t) \geq 1 - r$ and $M(b, x, t) \geq 1 - r$. Take $k \in (0,1)$ and setz $= S(a, b, k)$. By (2),

$$M(z, x, t) > min\{M(a, x, t), M(b, x, t)\} \geq 1 - r,$$

so $z \in B(x, r, t)$. For $k = 0$ or $k = 1$ we have $S(a, b, 0) = b$ and $S(a, b, 1) = a \in B(x, r, t)$, so the ball is convex.

5. COMMON FIXED POINT THEOREM FOR TWO WEAKLY COMPATIBLE MAPPINGS

We now prove the main common fixed point theorem for a pair of non-expansive mappings defined on compact convex subsets of a strictly convex fuzzy metric space. For this purpose, we use the following form of Zorn's lemma.

Lemma 5.1 Let $(P, \leq)$ *be a non-empty partially ordered set in which every chain has a lower bound. Then P has a minimal element.*

Theorem 5.2. *Assume that* $(X, M, *, S)$ *is a strictly convex fuzzy metric space satisfying (2). If* K*is a non-empty compact convex subset of* X *and* $f, g : K \to K$ *are non-expansive mappings that are weakly compatible, then* f *and* g *possess a common fixed point in* K.

Proof. Let F be the collection of all non-empty, closed and convex subsets $A \subset K$ such that

$f(A) \subset A$ *and* $g(A) \subset A$.

Note that $\mathcal{F}$ is non-empty, since $K \in \mathcal{F}$.

Partially order $\mathcal{F}$ by inclusion. Let $\{A_i : i \in I\} \subset \mathcal{F}$ be a chain. Then

$$A = \bigcap_{i \epsilon I} A_i$$

is non-empty (otherwise the closed sets Ai would contradict compactness of K), closed and convex, and clearly $f(A) \subset A$ and $g(A) \subset A$. Thus A is a lower bound for the chain. By Zorn's lemma, there exists a minimal element $K_0 \in \mathcal{F}$.

We claim that K0 consists of a single point. Suppose on the contrary that K_0 contains at least two distinct points. Then, by Lemma 4.1, K_0 possesses a normal structure, since it is compact (as a closed subset of K) and convex. Hence there exists a non-diametral point $x_0 \epsilon K_0$, so there is $t_0 > 0$ such that

$$inf_{y \in K_0} sup_{0<\varepsilon<t_0} M(x_0, y, \varepsilon) > d_{K_0}(t_0)$$

set

$$1 - \eta := inf_{y \epsilon K_0} sup_{0<\varepsilon<t_0} M(x_0 , y, \varepsilon),$$

So $1-\eta > d_{K_0}(t_0)$.

Let K_1 be the closed convex hull of $(K_0) \cup g(K_0)$, that is

$$K_1 = conv(f(K_0) \cup g(K_0)) .$$

Since $f(K_0) \cup g(K_0) \subset K_0$ and K_0 is closed and convex, we have $K_1 \subset K_0$. Also, $f(K_1) \cup g(K_1) \subset f(K_0) \cup g(K_0) \subset K_1$, so $K_1 \in \mathcal{F}$. By minimality of K_0 we obtain $K_1 = K_0$.

Now consider the family of closed balls

$C := \bigcap_{y\epsilon K_0} B(y,\eta,t_0),\ C := \bigcap_{y\epsilon f(K_0)\cup g(K_0)} B(y,\eta,t_0),$

where

$$B(y,\eta,t_0) = \{z \in X : M(y,z,t_0) \geq 1-\eta\}$$

by construction, $x_0 \in C$, since for all $y \epsilon K_0$ we have

$$sup_{0<\varepsilon<t_0} M(x_0,y,\varepsilon) \geq 1-\eta,$$

which implies $M(x_0,y,t_0) \geq 1-\eta$ by monotonicity in t. Hence C is non-empty.

Using Lemma 4.2 and the fact that intersections of closed convex sets are closed and convex, both C and C_1 are closed and convex. Since $f(K_0) \cup g(K_0) \subset K_0$, it is clear that $C_1 \subset C$.

We claim that $f(C) \subset C_1$ and $g(C) \subset C_1$. Let $z \epsilon C$ and $y \in f(K_0)$. There exists $x \in K_0$ such that $y = f(x)$. Since $z \epsilon C$, $M(x,z,t_0) \geq 1-\eta$. As f is non-expansive, we have

$M(f(z),y,t_0) = M(f(z),f(x),t_0) \geq M(z,x,t_0) \geq 1-\eta,$

so, $f(z) \in B(y,\eta,t_0)$ for all $y \in f(K_0)$. A similar argument with g shows that $g(z) \in B(y,\eta,t_0)$ for all $y \in g(K_0)$. Thus $f(C) \subset C_1$ and $g(C) \subset C_1$.

Since $C_1 \subset C \subset K_0$ and C is non-empty, closed and convex with *f(C)* $\subset C$ and *g(C)* $\subset C$, we have $C \epsilon \mathcal{F}$. By minimality of K_0 we must have $C = K_0$.

However, by construction, $C \subset K_0$ and for all $z \epsilon C$,

$$M(z,y,t_0) \geq 1-\eta \text{ for all } y \in K_0,$$

hence

$$d_C(t_0) \geq 1-\eta > d_{K_0}(t_0).$$

Since $C = K_0$, this gives $d_{K_0}(t_0) > d_{K_0}(t_0)$, a contradiction. Therefore K0 must consist of a single point, say $K_0 = \{x^*\}$.

Because $f(K_0) \subset K_0$ and $g(K_0) \subset K_0$, we have $f(x^*) \in \{x^*\}$ and $g(x^*) \in \{x^*\}$, so $f(x^*) = x^*$ and $g(x^*) = x^*$. Thus x^* is a common fixed point of *f* and *g*, which completes the proof.

If we put $g = f$ in Theorem 5.2, then we have the following result:

Corollary 5.3. *Let* $(X, M, *, S)$ *be a strictly convex fuzzy metric space satisfying (2), and let* K *be a non-empty compact convex subset of* X. If $f\colon K \to K$ *is a non-expansive mapping, then* f *has a fixed point in* K.

Since commuting mappings are also weakly compatible, in view of in Theorem 5.2, we have the following result:

Corollary 5.4. *Let* $(X, M, *, S)$ *be a strictly convex fuzzy metric space satisfying (2). Let* K *be a non-empty compact convex subset of* X. *If* $f, g : K \to K$ *are commuting non-expansive mappings, then* f *and* g *have a common fixed point in* K.

CONCLUSION

In this paper, we established a common fixed point theorem for a pair of weakly compatible non-expansive mappings defined on compact convex subsets of strictly convex fuzzy metric spaces satisfying a convexity condition. The result extends Takahashis [5] fixed point theorem from convex metric spaces to the framework of fuzzy metric spaces. It also generalizes earlier fixed point results for a single mapping and provides a common fixed point result for two mappings in this setting.

REFERENCES

1. Browder, F. E. (1965). Nonexpansive nonlinear operators in a Banach space. *Proceedings of the National Academy of Sciences*, *54*(4), 1041-1044.
2. George, A., & Veeramani, P. (1994). On some results in fuzzy metric spaces. *Fuzzy sets and systems*, *64*(3), 395-399.
3. Ješić, S. N., & Babačev, N. A. (2008). Common fixed point theorems in intuitionistic fuzzy metric spaces and L-fuzzy metric spaces with nonlinear contractive condition. *Chaos, Solitons and Fractals*, *37*(3), 675-687.
4. Kirk, W. A. (1965). A fixed point theorem for mappings which do not increase distances. *The American Mathematical Monthly*, *72*(9), 1004-1006.
5. Takahashi, W. (1969). Fixed point theorem for amenable semigroup of non-expansive mappings. In *Kodai Mathematical Seminar Reports,* 21(4),383-386. Department of Mathematics, Tokyo Institute of Technology.