ISSN: 2178-7727

Investigating Distance Magic Labeling In Mycielskian Graphs: Properties And Patterns

[1]Dr. Arunkumar. N. Yalal, [2]Dr. Ashok Vangeri

[1]Professor Department of Mathematics, Basavakalyan Engineering college, Basavakalyan
[2]Professor Mech Engg Dept. and Principal, Basavakalyan Engineering College, Basavakalyan

ABSTRACT

Distance magic labeling is a fascinating topic within graph theory that assigns distinct integers to the vertices of a graph such that the sum of labels of all vertices at a fixed distance from a given vertex is constant. This paper investigates the application of distance magic labeling to Mycielskian graphs, a class of graphs constructed to increase chromatic number without introducing additional cliques. The study delves into the unique properties of Mycielskian graphs and explores the existence and conditions under which they exhibit distance magic labeling. Through theoretical analysis and pattern recognition, we identify key characteristics of Mycielskian graphs that make them suitable for distance magic labeling. Furthermore, we discuss algorithmic approaches for efficiently determining such labelings and present illustrative examples to highlight critical results. The findings contribute to a deeper understanding of labeling in Mycielskian graph structures and open new avenues for further research in graph labeling theories and applications.

KEYWORDS: Distance Magic Labeling, Graph Properties, Combinatorics, Vertex Labeling, Graph Patterns

INTRODUCTION

Graph labeling, a prominent area within graph theory, involves assigning labels, typically integers, to the elements of a graph—vertices, edges, or both—while adhering to certain rules or constraints. Among the numerous graph labeling techniques, distance magic labeling has garnered significant interest due to its intriguing mathematical properties and potential applications in network design, coding theory, and communication systems. In distance magic labeling, distinct integers are assigned to the vertices of a graph in such a way that for every vertex, the sum of the labels of its neighbors at a fixed distance remains constant. This constant is often referred to as the "magic constant." Mycielskian graphs are a special class of graphs constructed from an existing graph to increase the chromatic number while avoiding the creation of new cliques. Initially introduced by Jan Mycielski to provide examples of triangle-free graphs with arbitrarily high chromatic numbers, Mycielskian graphs have become a rich subject of study in the context of graph coloring and structural properties. These graphs exhibit a complex structure, making them an interesting candidate for studying distance magic labeling

ISSN: 2178-7727

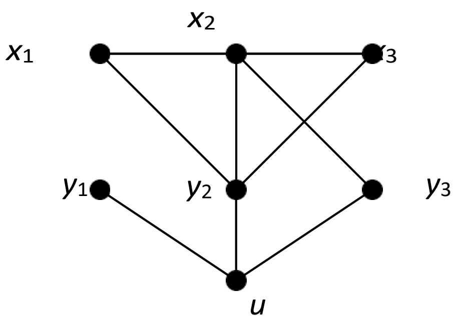


Figure 1. Mycielskian of P_3.

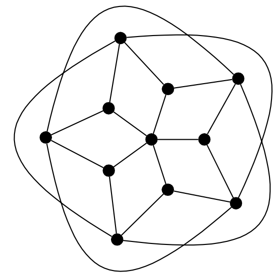


Figure 2. $\mu(C_5)$: Grötzsch Graph

The construction preserves the property of being triangle-free but increases the chromatic num-ber; by applying the construction repeatedly to a triangle-free starting graph, Mycielski showed that there exist triangle-free graphs with an arbitrarily large chromatic number. For example, starting with the graph $G = K_2$, which is triangle-free with $\chi(G) = 2$, we obtain $\mu(G) = C_5$ a cycle on5 vertices and $\chi(C_5) = 3$. Further $\mu^2(G) = \mu(\mu(G)) = \mu(C_5)$ is a Grötzsch graph (see Figure 2 with the chromatic number 4 and so on. We define $\mu^r(G) \cong \mu(\mu^{r-1}(G))$ for $r \geq 1$.

In this paper, we investigate whether there exists distance magic labeling of Mycielskian of various families of graphs.

Observe that, for a connected graph G with $|V(G)| = n$ and $|E(G)| = m$, $\mu(G)$ is also connected with $|V(\mu(G))| = 2n + 1$, and $|E(\mu(G))| = 3m + n$. Though there are other interesting structural relations between G and $\mu(G)$ such as edge connectivity, vertex connectivity, etc., weare not proving them here as they are beyond the interest of this article.

Assume that

$$\sum_{i=1}^{m} f(x_{1i}) = \alpha, \quad \sum_{j=1}^{n} f(x_{2j}) = \beta, \quad \sum_{i=1}^{m} f(y_{1i}) = \gamma, \quad \sum_{j=1}^{n} f(y_{2j}) = \delta. \tag{1}$$

Then the weights of the vertices are as follows:

$$w(x_{1i}) = \sum_{j=1}^{n} f(x_{2j}) + \sum_{j=1}^{n} f(y_{2j}) = \beta + \delta \text{ for each } 1 \leq i \leq m$$

$$w(x_{2j}) = \sum_{i=1} f(x_{1i}) + \sum_{i=1} f(y_{1i}) = \alpha + \gamma \text{ for each } 1 \leq j \leq n$$

$$w(y_{1i}) = \quad f(x_{2j}) + f(u) = \beta + f(u) \text{ for each } 1 \leq i \leq m$$

$$w(y_{2j}) = \quad f(x_{1i}) + f(u) = \alpha + f(u) \text{ for each } 1 \leq j \leq n$$

$$w(u) = \sum_{i=1}^{m} f(y_{1i}) + \sum_{j=1}^{n} f(y_{2j}) = \gamma + \delta.$$

Since, the Mycielskian graph $\mu(G)$ is distance magic, the vertex weights are the same under f
i.e.

$$\beta + \delta = \alpha + \gamma = \beta + f(u) = \alpha + f(u) = \gamma + \delta.$$

From the above equations we get,

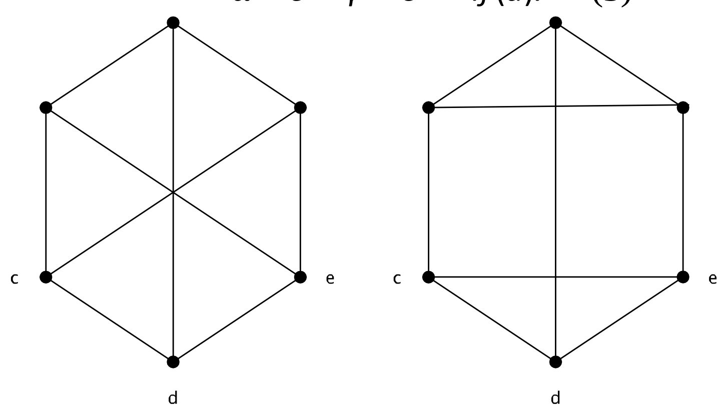
$$\alpha = \beta = \gamma = \delta = f(u). \qquad (2)$$

The vertex u must receive the largest label, that is, $f(u) = 2(m + n) + 1$. Otherwise, one ofthe vertex x_{1i}, x_{2i}, y_{1j} or y_{2j} for some i or j will receive the label $2(m + n) + 1$ and one of theequalities

$$\alpha = f(u), \ \beta = f(u), \ \gamma = f(u), \ \delta = f(u)$$

is not possible. Therefore, from Equation (13) we have

$$\alpha + \beta + \gamma + \delta = 4f(u). \qquad (3)$$



(a) $K_{3,3}$ (b) G

Figure 3-regular graphs of order 6.

Lemma 1. *The Mycielskian of a* 3*-regular graph of order* 8 *is not distance magic.*

Proof. Since, we know that there are five 3-regular graphs of order 8 [3], denoted by G_1, G_2, G_3,

ISSN: 2178-7727

G_4, G_5 as shown in Figure 3 To apply Lemma 1 for each of these graphs we identify two vertices u and v in each graph to get 2 as the size of symmetric difference $N(u) \triangle N(v)$ as follows:

1. In graph G_1, $N(a) \triangle N(g) = \{b, d\}$ and hence $|N(a) \triangle N(g)| = 2$.
2. In graph G_2, $N(a) \triangle N(b) = \{a, b\}$ and hence $|N(a) \triangle N(b)| = 2$.
3. In graph G_3, $N(b) \triangle N(g) = \{a, f\}$ and hence $|N(b) \triangle N(g)| = 2$.
4. In graph G_4, $N(c) \triangle N(g) = \{b, h\}$ and hence $|N(c) \triangle N(g)| = 2$.
5. In graph G_5, $N(a) \triangle N(d) = \{b, c\}$ and hence $|N(a) \triangle N(d)| = 2$.

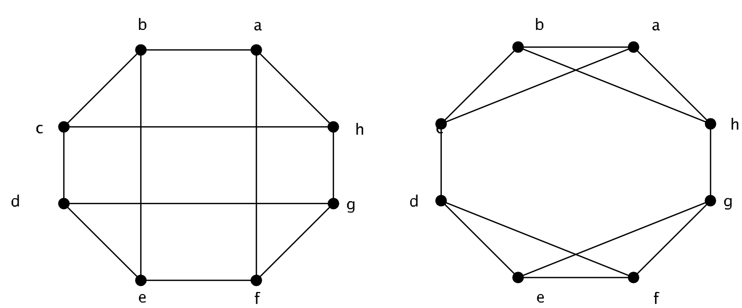
Then by Lemma 1, Mycielskian of a 3-regular graph of order 8 is not distance magic.

Theorem 1. *The Mycielskian of* 3*-regular graph* G *of order* ≤ 8 *is not distance magic.*

There are nineteen 3-regular graphs of order 10 [3]. We consider the Petersen graph—thebest-known graph in this family.

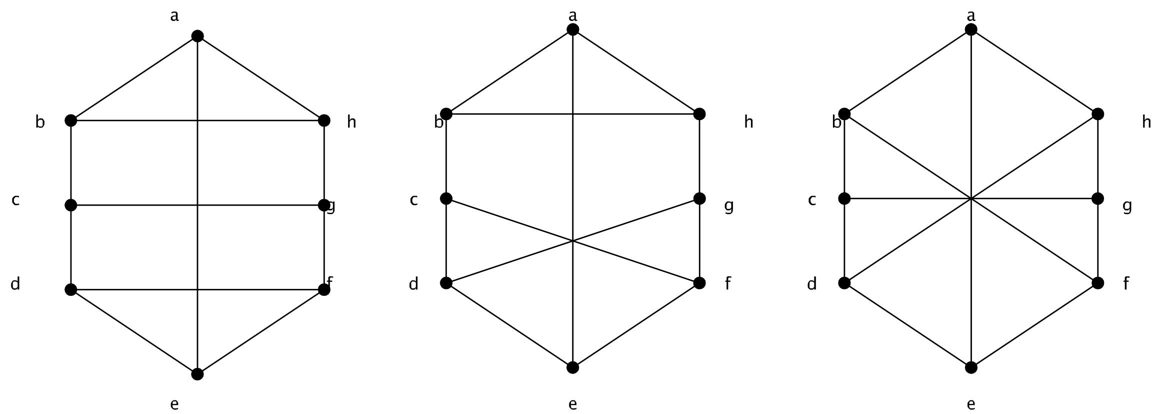
Theorem 2. *The Mycielskian of the Petersen graph is not distance magic.*

Proof. Let G denote the Petersen graph as shown in Figure 4. On contrary suppose that Myciel- skian of Petersen admits distance magic labeling f. Then the neighborhoods of vertices y_1, y_4, y_7, $y_8$in $\mu(G)$ as shown in Figure 4 are:



(a) G_1 (b) G_2

DOI: https://doi.org/10.57030/ASCI.25.4.AS23



(c) G_3 (d) G_4 (e) G_5

Figure 4-regular graphs of order 8.

$$N_{\mu(G)}(y_1) = \{x_2,\ x_5,\ x_6,\ u\}$$
$$N_{\mu(G)}(y_4) = \{x_3,\ x_5,\ x_9,\ u\}$$
$$N_{\mu(G)}(y_7) = \{x_2,\ x_9,\ x_{10},\ u\}$$
$$N_{\mu(G)}(y_8) = \{x_3,\ x_6,\ x_{10},\ u\}.$$

Hence, their weights are given by,

$$w(y_1) = f(x_2) + f(x_5) + f(x_6) + f(u)$$
$$w(y_4) = f(x_3) + f(x_5) + f(x_9) + f(u)$$
$$w(y_7) = f(x_2) + f(x_9) + f(x_{10}) + f(u)$$
$$w(y_8) = f(x_3) + f(x_6) + f(x_{10}) + f(u).$$

Since, all weights are same, $w(y_1) = w(y_7)$ gives

$$f(x_5) + f(x_6) = f(x_9) + f(x_{10}) \qquad (4)$$

and $w(y_4) = w(y_8)$ gives

$$f(x_5) + f(x_9) = f(x_6) + f(x_{10}). \quad (5)$$

Subtracting Equation (4) from Equation (5) we obtain a contradiction $f(x_6) = f(x_9)$. □

ISSN: 2178-7727

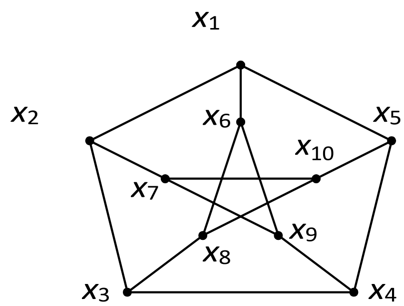


Figure 5. Petersen Graph.

Proposition 1. *Let G be a graph. If $\mu(G)$ is a regular graph, then $\mu(G)$ is not distance magic.*

Proof. Let G be a graph on n vertices such that $\mu(G)$ is r-regular. Then by Theorem 1, $G \cong K_2$. But $\mu(K_2) \cong C_5$ and by Theorem 2, C_5 is not distance magic. This completes the proof. $\square$

Observation 1. The graph G and its Mycielskian $\mu(G)$ do not share the property of being distance magic, i.e. $\mu(G)$ is distance magic irrespective of G, e.g., the path on 3 vertices P_3 is distance magic [10] but $\mu^r(P_3)$ is not distance magic for any $r \geq 1$ (see Corollary 1). Whereas, C_4 is distance magic [10] and $\mu(C_4)$ is also distance magic (see Theorem 2) but $\mu^2(C_4)$ is not distance magic.

CONCLUSION AND FUTURE SCOPE

It remains to find other classes of graphs whose Mycielskian is distance magic. To construct distance magic graphs of arbitrarily large chromatic number by Mycielskians construction we needa graph G such that $\mu^r(G)$ is distance magic, for all $r \geq 1$ but Observation 1, makes it hard to thinkof such a graph G

REFERENCES

1. Gallian, J. A. (2018). A Dynamic Survey of Graph Labeling. The Electronic Journal of Combinatorics, 5, DS6.
2. Chartrand, G., Lesniak, L., & Zhang, P. (2010). Graphs & Digraphs (5th ed.). CRC Press.
3. Mycielski, J. (1955). Sur le coloriage des graphes. Colloquium Mathematicae, 3(161), 161–162.
4. Slamin, & Suprijanto, D. (2010). Distance Magic Labeling of Graphs. Bulletin of the Malaysian Mathematical Sciences Society, 33(1), 17–25.
5. Slamin, Miller, M., & Bačáková, Z. (2014). On Distance Magic and Antimagic Labelings of Graphs. Discussiones Mathematicae Graph Theory, 34(4), 783–794.
6. Cameron, K., & Daigle, R. (2018). Magic Labelings of Graphs: A Survey and Open Problems. Discrete Mathematics, 341(7), 1891–1903.
7. West, D. B. (2001). Introduction to Graph Theory (2nd ed.). Prentice Hal
8. Harary, F. (1969). Graph Theory. Addison-Wesley Publishing Company.
9. Yebra, J. L. A., Fiol, M. A., & Fàbrega, J. (1984). Distance Magic Labeling in Certain Graphs. Ars Combinatoria, 17(A), 89–98.

10. 10. Gutin, G., & Bang-Jensen, J. (2001). Digraphs: Theory, Algorithms and Applications. Springer.

ISSN: 2178-7727